Lecture 9, 6/16/2017

• Example: Collisions in Two Dimensions
• Center of Mass (CM)
• Angular Quantities
• Constant Angular Acceleration
• Rolling Motion (Without Slipping)
• Vector Nature of Angular Quantities
Collisions in Two Dimensions

Conservation of energy and momentum can also be used to analyze collisions in two (or three) dimensions.
A proton \((m_p = 9.0 \times 10^{-31} \text{ kg})\) collides elastically with another proton which is initially at rest. The incoming proton has a speed of \(3.5 \times 10^5 \text{ m/s}\) and makes a glancing collision with the second proton. After the collision, one proton moves off at an angle of \(37^\circ\) to the original direction of motion, and the second deflects at an angle of \(53^\circ\). What are the final speeds of the two protons?
Center of Mass

In (a), the diver’s motion is pure translation; in (b) it is translation plus rotation.

There is one point that moves in the same path a particle would take if subjected to the same force as the diver. This point is called the center of mass (CM).
Center of Mass

The general motion of an object can be considered as the sum of the translational motion of the CM, plus rotational, vibrational, or other forms of motion about the CM.
Center of Mass

For two particles, the center of mass lies closer to the one with the most mass:

\[ x_{\text{CM}} = \frac{m_A x_A + m_B x_B}{m_A + m_B} = \frac{m_A x_A + m_B x_B}{M}, \]

where \( M \) is the total mass.

\[ v_{\text{CM}} = \frac{m_A v_A + m_B v_B}{M} \]
Center of Mass vs. Center of Gravity

The center of gravity is the point where the gravitational force can be considered to act. **It is the same as the center of mass as long as the gravitational force does not vary among different parts of the object.**
Center of Mass

The center of gravity can be found experimentally by suspending an object from different points. The CM need not be within the actual object—a doughnut’s CM is in the center of the hole.
The location of the center of mass of the leg (circled) will depend on the position of the leg.
CM for the Human Body

High jumpers have developed a technique where their CM actually passes under the bar as they go over it. This allows them to clear higher bars.
Center of Mass and Translational Motion

The CM (which may not correspond to the position of any particle) continues to move according to the net force.
Rotational Motion
Angular Quantities

In purely rotational motion, all points on the object move in circles around the axis of rotation ("O").

The radius of the circle is $r$. All points on a straight line drawn through the axis move through the same angle in the same time.

The angle $\theta$ in radians is:

$$\theta = \frac{l}{r},$$

where $l$ is the arc length.
Angular Quantities

Angular displacement:

\[ \Delta \theta = \theta_2 - \theta_1 \]

The average angular velocity is defined as the total angular displacement divided by time:

\[ \bar{\omega} = \frac{\Delta \theta}{\Delta t}, \]

The instantaneous angular velocity:

\[ \omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t}. \]
Angular Quantities

The angular acceleration is the rate at which the angular velocity changes with time:

\[ \alpha = \frac{\omega_2 - \omega_1}{\Delta t} = \frac{\Delta \omega}{\Delta t} \]

The instantaneous acceleration:

\[ \alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} \]
Angular Quantities

Every point on a rotating body has an angular velocity \( \omega \) and a linear velocity \( v \).

They are related:

\[
v = \frac{\Delta l}{\Delta t} = r \frac{\Delta \theta}{\Delta t}
\]

or (since \( \Delta \theta/\Delta t = \omega \))

\[
v = r\omega.
\]
Angular Quantities

Therefore, objects farther from the axis of rotation will move faster.
Tangential and Centripetal Acceleration

If the angular velocity of a rotating object changes, it has a tangential acceleration:

\[
a_{\text{tan}} = \frac{\Delta v}{\Delta t} = r \frac{\Delta \omega}{\Delta t} = r \alpha
\]

Even if the angular velocity is constant, each point on the object has a centripetal acceleration:

\[
a_c = \frac{v^2}{r} = \frac{(r \omega)^2}{r} = \omega^2 r
\]
Angular Quantities

Here is the correspondence between linear and rotational quantities:

<table>
<thead>
<tr>
<th>Linear</th>
<th>Type</th>
<th>Rotational</th>
<th>Relation‡</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>displacement</td>
<td>$\theta$</td>
<td>$x = r\theta$</td>
</tr>
<tr>
<td>$v$</td>
<td>velocity</td>
<td>$\omega$</td>
<td>$v = r\omega$</td>
</tr>
<tr>
<td>$a_{\text{tan}}$</td>
<td>acceleration</td>
<td>$\alpha$</td>
<td>$a_{\text{tan}} = r\alpha$</td>
</tr>
</tbody>
</table>

‡ You must use radians.
Angular Quantities

**Frequency** is the number of complete revolutions per second:

\[ f = \frac{\omega}{2\pi} \]

Frequencies are measured in hertz.

\[ 1 \text{ Hz} = 1 \text{ s}^{-1} \]

The period is the time one revolution takes. It is related to frequency:

\[ T = \frac{1}{f} \]
# Constant Angular Acceleration

The equations of motion for constant angular acceleration are the same as those for linear motion, with the substitution of the angular quantities for the linear ones.

<table>
<thead>
<tr>
<th>Angular</th>
<th>Linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega = \omega_0 + \alpha t )</td>
<td>( v = v_0 + at )</td>
</tr>
<tr>
<td>( \theta = \omega_0 t + \frac{1}{2} \alpha t^2 )</td>
<td>( x = v_0 t + \frac{1}{2} at^2 )</td>
</tr>
<tr>
<td>( \omega^2 = \omega_0^2 + 2 \alpha \theta )</td>
<td>( v^2 = v_0^2 + 2ax )</td>
</tr>
<tr>
<td>( \overline{\omega} = \frac{\omega + \omega_0}{2} )</td>
<td>( \overline{v} = \frac{v + v_0}{2} )</td>
</tr>
</tbody>
</table>

[constant \( \alpha, a \) ]

[constant \( \alpha, a \) ]

[constant \( \alpha, a \) ]

[constant \( \alpha, a \) ]
Rolling Motion (Without Slipping)

In (a), a wheel is rolling without slipping. The point P, touching the ground, is instantaneously at rest, and the center moves with velocity v.

In (b) the same wheel is seen from a reference frame where C is at rest. Now point P is moving with velocity \(-v\).

Relationship between linear and angular speeds: \( v = r\omega \)
Vector Nature of Angular Quantities

The angular velocity vector points along the axis of rotation; its direction is found using a right hand rule: